AD-A107 828   ROCHESTER UNIV N Y DEPT OF ELECTRICAL ENGINEERING   F/G 12/1
IMPULSE RESPONSE DETERMINATION IN THE TIME DOMAIN.(U)
NOV 81   T K SARKAR, D D WEINER, V K JAIN   N00014-79-C-0598
UNCLASSIFIED   TR-81-10   NL

1 OF 1
AD A
107828

END
DATE
FILMED
1-82
DTIC

LEVEL

(10)

AD A107828

IMPULSE RESPONSE DETERMINATION IN THE TIME DOMAIN

Tapan K. Sarkar[1]

Donald D. Weiner[2]

Vijay K. Jain[3]

Soheil A. Dianat[1]

This work has been supported by the Office of Naval Research under contract N00014-79-C-0598.

1. Dept. of Electrical Engr.
   Rochester Inst. of Tech.
   Rochester, NY 14623
2. Dept. of Electrical Engr.
   Syracuse University
   Syracuse, NY 13210
3. Department of Electrical Engr.
   University of South Florida
   Tampa, Florida 33620

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TR-81-10	2. GOVT ACCESSION NO. [illegible]	3. RECIPIENT'S CATALOG NUMBER [illegible]
4. TITLE (and Subtitle) Impulse Response Determination in the Time Domain		5. TYPE OF REPORT & PERIOD COVERED Technical Report 10
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Tapan K. Sarkar, Soheil A. Dianat, Vijay K. Jain, Dona d D. Weiner		8. CONTRACT OR GRANT NUMBER(s) N00014-79-C-0598
9. PERFORMING ORGANIZATION NAME AND ADDRESS Dept. of Electrical Engineering Rochester Institute of Technology Rochester, New York 14623		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61153N RR021-01 RR021-01-01 NR371-014
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, Virginia 22217		12. REPORT DATE November 1981
		13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

[illegible]

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Impulse Response
System Identification
Time Domain

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper two methods are presented for determining the impulse response of an object in the time domain when both the input and output time domain waveforms are specified. Pole extraction features can then be applied to the nonimpulsive portion of the impulse response to determine the SEM parameters of the object. The first method involves synthetic division and is comparatively straightforward The second technique is a least squares method which is computationally more stable. The effect of measurement errors in the input and output waveforms is evaluated for each method. An investigation is made as to what form of the input

20.

minimizes the noise variances in the computation. Finally a generalized least squares technique is presented which yields a minimum variance unbiased estimate for the impulse response when the noise covariance matrix is known.

1. BACKGROUND

In order to characterize an unknown object by remote sensing, certain electrical properties of the object may be used [1-9]. In particular, the impulse response, ramp response and the natural resonant frequencies of the object have been proposed. In order to accurately probe the object, one must illuminate it with a band of frequencies whose wavelengths are approximately the same dimensions as the overall length of the object. Since, in general, the size of the object is not known beforehand, it is necessary to illuminate the object by a broadband signal such as a waveform which is an approximation to an impulse. The transmitted pulse induces electrical currents on the object and reradiates the incidental energy. The problem is then to obtain the electrical properties of the object from the transmitted and received time domain waveforms. This area has been investigated by the researchers at the Ohio State University [1-5].

In recent years with the advent of the singularity expansion method, efforts have been directed towards characterizing an object by its natural frequencies - poles and zeros [6-8]. For this purpose, Prony's method has been applied with much success in analyzing measured impulse responses with high signal to noise ratio. However, it is difficult to extend Prony's method to arbitrary input and arbitrary output waveforms. Also the SEM representation does not account for the impulsive portion of the system time behavior. Hence, the complete impulse response contains more information than the SEM poles. One of the objectives of this paper is to obtain the impulse response in the time domain when arbitrary input and output waveforms are given.

More recently, the pencil of function method [10-14] has been applied with much success to obtain the natural frequencies of objects from measured arbitrary input and output time waveforms. Both Prony's and the pencil of function method are frequency domain techniques. The accuracies of determining impulse responses by either of these techniques depend directly on the time domain data available. Also a significant problem with any such parametric scheme is the need to estimate the number of dominant poles and zeros. A poor estimate can lead to large errors in the results.

Finally as discussed in the next section, it is not easy to obtain the impulse response from given input and output time domain waveforms using Fast Fourier Transform techniques [9].

The objective of this paper is to present two techniques for obtaining the impulse response of an object directly in the time domain from measured input and output waveforms.

2. INTRODUCTION

Let x(t) be the input to a time invariant causal linear system which is characterized by its impulse response h(t). The corresponding output y(t) is given by

$$y(t) = \int_0^\infty x(t-\tau)h(\tau)\, d\tau \qquad (1)$$

One approach for solving (1) for the impulse response involves the Laplace Transform. The result is

$$Y(s) = H(s)\cdot X(s) \qquad (2)$$

where X(s), Y(s) and H(s) are the Laplace Transforms of x(t), y(t) and h(t), respectively. The impulse response h(t) is obtained by taking the inverse Laplace Transform of (2) to yield

$$h(t) = \frac{1}{2\pi j} \int_\Gamma \frac{Y(s)}{X(s)} \exp(st)\, ds \qquad (3)$$

where Γ is the Bromwich contour. Numerical problems are frequently encountered with this approach when dealing with real data [9]. For example, when using Fast Fourier Transform techniques, the division blows up if $X(jw)=0$ in the pass band.

Two time domain methods are presented in this paper. One is the method of synthetic division while the other is the method of least squares. The former is computationally straightforward whereas the latter is computationally more stable.

In this presentation $y(t)$ and $x(t)$ are assumed to be discrete time signals with identical sampling rates. The problem is to process the discrete data points so as to estimate the corresponding discrete impulse response without introducing additional information like the augmented high frequency response [9].

Let the ith samples of $y(t)$, $x(t)$ and $h(t)$ be denoted by y_i, x_i, and h_i, respectively. Also, assume that $y(t)$ and $x(t)$ are available as the finite time series $\{y\} = \{y_0, y_1, y_2, \ldots, y_N\}$ and $\{x\} = \{x_0, x_1, x_2, \ldots, x_M\}$. The object is to estimate the finite impulse response time series $\{h\} = \{h_0, h_1, h_2, \ldots, h_L\}$.

For convenience, the following assumptions are made:

(a) $h_i = 0$ for $i > L$.

(b) x_j is observed for $0 \leq j \leq M$ and is not identically zero.

(c) $x_j = 0$ for $0 > j$ and $j > M$.

(d) y_k is observed for $0 \leq k \leq N$ $(=M+L)$

3. BA HLI'S TECHNIQUE (SYNTHETIC DIVISION) [15-16]

This method is applicable for all positive integer choices of N, M and L. The sequence corresponding to $h(t)$ is written symbolically as

$$\{h\} = \frac{\{y\}}{\{x\}} = \frac{\{y_0, y_1, y_2, \ldots, y_N\}}{\{x_0, x_1, x_2, \ldots, x_M\}} = \{h_0, h_1, \ldots, h_L\} \qquad (4)$$

where the division of sequence {y} by sequence {x} is carried out by synthetic division.

Justification of the synthetic division operation is obtained from the theory of z-transforms. The input and output z-transforms can be written as

$$y(z) = y_0+y_1z^{-1}+y_2z^{-2}+\ldots+y_Nz^{-N} \tag{5}$$

$$x(z) = x_0+x_1z^{-1}+x_2z^{-2}+\ldots+x_Mz^{-M}. \tag{6}$$

h(z) is then the ratio of the two polynomials given by (5) and (6). Since two z-polynomials are divided in the same manner as algebraic polynomials, the operation of synthetic division is justified.

Alternatively, a *discrete* approximation to (1) is

$$y_j = \Delta\tau \sum_{i=0}^{j} h_{j-i}x_i \tag{7}$$

where $\Delta\tau$ is the sampling interval. A computationally compact form for h_j is then given by

$$h_j = \frac{y_j-\Delta\tau \sum_{i=1}^{j} x_i h_{j-i}}{x_0\Delta\tau} \tag{8}$$

where it is assumed $x_0 \neq 0$. The values for h_j obtained in this way differ from those obtained by synthetic division by the constant factor $1/\Delta\tau$.

Note that both of these methods are applicable to discontinuous inputs and outputs.

Example 1: As a simple example, consider the rectangular pulse {x} = {1, 1, 1} to be an input to a system whose output is the triangular pulse {y} = {1,2,3,2,1}. Using synthetic division, as implied by (4) the unknown transfer function {h} is given by

$$h(z) = \frac{1+2z^{-1}+3z^{-2}+2z^{-3}+z^{-4}}{1+z^{-1}+z^{-2}} = 1 + z^{-1} + z^{-2} \tag{9}$$

This implies {h} = {1,1,1}.

Alternatively, if it is assumed $\Delta\tau = 1$, application of (8) results in {h} = {1,1,1}. This agrees with the solution obtained by synthetic division.

4. EFFECT OF MEASUREMENT ERRORS

The effect of measurement errors on the synthetic division scheme is now considered. Assume that the input measurement is corrupted by additive noise p(t) so that the total recorded input is x+p. Also assume that the output measurement is corrupted by noise q(t) such that the total measured output is y+q. Using samples of the corrupted input and output measurements, an estimate of the impulse response, denoted by $\tilde{h}(z)$, is given by

$$\tilde{h}(z) = \frac{y(z)+q(z)}{x(z)+p(z)} = \frac{(y_0+y_1z^{-1}+y_2z^{-2}+\ldots+y_Nz^{-N})+(q_0+q_1z^{-1}+\ldots+q_Nz^{-N})}{(x_0+x_1z^{-1}+\ldots+x_Mz^{-M})+(p_0+p_1z^{-1}+\ldots+p_Mz^{-M})} \tag{10}$$

In this section the relationship of the estimate $\{\tilde{h}\}$ to the true impulse response $\{h\}$ is explored where it is assumed that the measurement errors are small and that the elements of $\{q\}$ and $\{p\}$ which pertain to two different measurements, are statistically independent zero mean random variables.

The expression in (10) can be rewritten as

$$\tilde{h}(z) = \frac{y(z)}{x(z)} \cdot \frac{1+\frac{q(z)}{y(z)}}{1+\frac{p(z)}{x(z)}} \tag{11}$$

For those values of z for which $\left|\frac{p(z)}{x(z)}\right| < 1$, $[1 + \frac{p(z)}{x(z)}]^{-1}$ may be expanded in a Binomial series to yield

$$[1 + \frac{p_0+p_1z^{-1}+\ldots+p_Mz^{M}}{x_0+x_1z^{-1}+\ldots+x_Mz^{-M}}]^{-1} = 1 - \frac{p(z)}{x(z)} + [\frac{p(z)}{x(z)}]^2 - \ldots. \tag{12}$$

It follows for these values of z that

$$\tilde{h}(z) = \frac{y(z)}{x(z)} \cdot [1 + \frac{q(z)}{y(z)}] \cdot [1 - \frac{p(z)}{x(z)} + (\frac{p(z)}{x(z)})^2 - \ldots] \tag{13}$$

The expected value of a random sequence is defined to be the sequence whose elements are expectations of the original elements. Since each element represents a sample value of the waveform, the expectation of the element corresponds to an ensemble average of the waveform at the instant of the sample value. Then the expected value of the estimate $\tilde{h}(z)$ is given by

$$E[\tilde{h}(z)] = \frac{y(z)}{x(z)} E[1+ \frac{q(z)}{y(z)}] \, E[1- \frac{p(z)}{x(z)} +\{ \frac{p(z)}{x(z)}\}^2 - \ldots]$$

$$= \frac{y(z)}{x(z)} \cdot [1+ \frac{1}{y(z)} E\{q(z)\}] \cdot [1- \frac{1}{x(z)} E\{p(z)\} + \frac{1}{[x(z)]^2} E\{p(z)\}^2 \ldots]$$

and (14)

$$E[q(z)] = E[q_0+q_1 z^{-1}+q_2 z^{-2}+\ldots]$$

$$= E[q_0]+E[q_1]z^{-1}+E[q_2]z^{-2}+\ldots$$

$$= 0 \tag{15}$$

where use has been made of the assumptions of zero mean and statistical independence of the random variables p_i and q_i. Similarly,

$$E\,[p(z)] = 0 \tag{16}$$

and

$$E[p(z)]^2 = E[p_0{}^2+2p_0p_1z^{-1}+2p_0p_2z^{-2}+\ldots$$
$$+p_1{}^2z^{-2}+2p_1p_2z^{-3}+\ldots+p_2z^{-4}+2p_2p_3z^{-5}+\ldots]$$
$$= \sigma_{p_0}{}^2 + \sigma_{p_1}{}^2 z^{-2} + \sigma_{p_2}{}^2 z^{-4} + \ldots \tag{17}$$

where

$$\sigma_{p_i}{}^2 = E[p_i^2]\,. \tag{18}$$

Consequently

$$E[\tilde{h}(z)] = h(z)\cdot[1+ \frac{\sigma_{p_0}{}^2 + \sigma_{p_1}{}^2 z^{-2} + \ldots}{(x_0+x_1z^{-1}+\ldots+x_M z^{-M})^2}]$$

$$= h'_0 + h'_1 z^{-1} + h'_2 z^{-2} + \ldots \tag{19}$$

where $h(z) = y(z)/x(z)$. Thus if the variances of the input measurement errors are known, approximate values for the expectation of the estimate of the time domain impulse response $\tilde{h}(z)$ can be evaluated. Notice that if $p_i \equiv 0$ for all i (i.e. the input measurement is free from any additive noise) then the estimate is unbiased even though a measurement error is still associated with the output. However if $p_i \not\equiv 0$ then there is a bias in the solution which is given as

$$E[\tilde{h}(z)] - h(z) = h(z) \cdot \left\{ \frac{\sigma_{p_0} + \sigma_{p_1}^2 z^{-2} + \ldots.}{(x_0 + x_1 z^{-1} + \ldots + x_M z^{-M})^2} \right\}$$

By definition, the variance of a sequence is the sequence whose elements are the variance of the original elements. Each variance is the variance of the original random waveform at the corresponding sampling instant. Note that

$$\tilde{h}(z) - E[\tilde{h}(z)] = h(z) \left[\frac{q(z)}{y(z)} - \frac{p(z)}{x(z)} + \left\{ \frac{p(z)}{x(z)} \right\}^2 - \frac{q(z)p(z)}{x(z)y(z)} - \frac{\sigma_p^2(z)}{[x(z)]^2} \right]$$

$$+ \ldots \text{ higher order terms} \qquad (20)$$

where $\sigma_p^2(z) = \sigma_{p_0}^2 + \sigma_{p_1}^2 z^{-2} + \sigma_{p_2}^2 z^{-4} + \ldots$

As a first approximation

$$\tilde{h}(z) - E[\tilde{h}(z)] \simeq \frac{q(z) - h(z)p(z)}{x(z)}$$

$$= c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots \qquad (21)$$

Note that (21) corresponds to the zero mean sequence $\{c_0, c_1, c_2, \ldots\}$. It follows that the variance of this sequence, which also equals the variance of $\tilde{h}(z)$, results in

$$\sigma^2[\tilde{h}(z)] = E[c_0^2] + E[c_1^2] z^{-1} + E[c_2^2] z^{-2} + \ldots$$

$$= \sigma_0^2 + \sigma_1^2 z^{-1} + \sigma_2^2 z^{-2} + \ldots \qquad (22)$$

where $\sigma_i^2 = E[c_i^2]$.

Use of (22) and (16), in conjunction with Tshebyshev's inequality [17], yields the inequality

$$P[h_i' - k\sigma_i < \hat{h}_i < h_i' + k\sigma_i] \geq 1 - \frac{1}{k^2} \qquad (23)$$

where P[.] denotes the probability of the argument in the brackets and $\hat{h}_i$ is the element in $\{\hat{h}\}$. Equation (23) is valid for small measurementerrors.

In summary, if the first and second order statistic of the random sequences {p} and {q} are known, then (22) and (16) yield approximate expressions for the variance and mean of the sample values of the extimate of the time domain impulse response obtained from (10).

<u>Example 2:</u> Consider a system with impulse response {h} = {1,1} for which the input is {x} = {1,1} and the output is {y} = {1,2,1}. Assume, for the purpose of illustration, that the input and output are each contaminated by zero mean additive measurement errors which take on the values +0.01 or -0.01 with equal probability. Note that the measurement errors have zero means and variances $\sigma^2 = 10^{-4}$. Consider a particular measurement error which results in the observations {x} = {1.01,0.99} and {y} = {0.99, 2.01, 1.0}. In this example, the effects of the measurement errors on the extimate {h} are explored.

From (10) the estimate of the impulse response is obtained as

$$\hat{h}(z) = \frac{\tilde{y}(z)}{\tilde{x}(z)} = \frac{.99+2.01z^{-1}+z^{-2}}{1.01+.99z^{-1}} = .9802+1.0293z^{-1}-.0190z^{-2}+.0190z^{-3} \qquad (24)$$

$$\{\hat{h}\} = \{.9802,\ 1.0293,\ -.0190,\ .0190,\ \ldots\} \qquad (25)$$

The measurement errors are seen to cause the elements in $\{\hat{h}\}$ to differ from those in {h} and to cause $\{\hat{h}\}$ to be an infinite sequence whereas {h} was finite.

By using (24) the expected value of the estimate is found to be

$$E[\hat{h}(z)] = (1+\sigma^2)+(1-\sigma^2)z^{-1}+2\sigma^2 z^{-2}-2\sigma^2 z^{-3}+\ldots$$

$$= 1.0001 + 0.9999z^{-1} + .0002z^{-2} - .0002z^{-3}+ \ldots \qquad (26)$$

It is seen that the mean value of the estimate is biased. This is due to measurement errors in the input.

The variances of the estimate are obtained from (21) and (22). Observe that

$$\hat{h}(z)-E[\hat{h}(z)] \approx \frac{q(z) - h(z)p(z)}{x(z)}$$

$$= (q_0-p_0)+(q_1-p_1-q_0)z^{-1}+(q_2-q_1+q_0)z^{-2}-(q_2+q_1+q_0)z^{-3}+ \ldots \qquad (27)$$

It follows that

$$\sigma^2[\hat{h}(z)] = [(q_0-p_0)^2]+ E[(q_1-p_1-q_0)^2]z^{-1}+E[(q_2-q_1+q_0)^2z^{-2}$$

$$+ E[(q_2-q_1+q_0)^2]z^{-3}+ \ldots$$

$$= \sigma^2[2+3z^{-1}+3z^{-2}+3z^{-3}+\ldots] \qquad (28)$$

Consequently the variance of the first element in $\{\hat{h}\}$ is $2\sigma^2$ while it is $3\sigma^2$ for the remaining elements.

Letting k=3 in (23), Tshebyshev's inequality yields

$$P[|\hat{h}_0-1.0001| \leq 3(.0141)] \geq 8/9$$

$$P[|\hat{h}_1-0.9999| \leq 3(.0173)] \geq 8/9$$

$$P[|\hat{h}_2- .0002| \leq 3(.0173)] \geq 8/9 \qquad (29)$$

Observe that the estimates in (25) are within the 3σ of the expected values.

A disadvantage with the synthetic division scheme is that the error tends to build up with the length of the sequence.

5. A LEAST SQUARES APPROACH TO THE ESTIMATION PROBLEM

If X(s) [which is the Laplace Transform of x(t)] contains a zero in the right half plane (RHP) [i.e. non minimum phase], a serious computational difficulty may arise with the synthetic division scheme because H(s) may then contain a pole in the RHP [18]. This results in numerical instability. With a stable system the RHP zero of X(s) is cancelled by the identical RHP in Y(s). However, in practice, round-off and/or truncation errors in computing X(s) and Y(s) prevent this cancellation. The synthetic division approach does not then converge. Should this occur, an alternate approach must be used. This section develops the least squares estimation of the time domain impulse response from measured input and output data.

With reference to (7), the response of a discrete time system may be written as

$$y'_j \triangleq \frac{y_j}{\Delta\tau} = \sum_{i=0}^{j} x_i h_{j-i} \tag{30}$$

The convolution in (30) may be expressed in matrix form as

$$\underset{NX1}{\begin{bmatrix} y'_0 \\ y'_1 \\ \cdot \\ \cdot \\ \cdot \\ y'_N \end{bmatrix}} \triangleq \underset{NXL}{\begin{bmatrix} x_0 & 0 & 0 & 0 & \dots & 0 \\ x_1 & x_0 & 0 & 0 & \dots & 0 \\ x_2 & x_1 & \cdot & \cdot & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_M & x_{M-1} & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & x_M \end{bmatrix}} \underset{LX1}{\begin{bmatrix} h_0 \\ h_1 \\ \cdot \\ \cdot \\ \cdot \\ h_L \end{bmatrix}}$$

$$[Y'] = [X][H] \tag{31}$$

where Y' and H are NX1 and LX1 column matrices, respectively, and X is an NXL rectangular matrix. Equation (31) can be solved for [H] by first premultiplying both sides by $[X]^T$ (i.e. transpose of [X]). Then

$$[X]^T[X][H] = [X]^T[Y'] \tag{32}$$

It follows that

$$[H] = \{[X]^T[X]\}^{-1}[X]^T[Y'] \quad (33)$$

It is interesting to note that $[X]^T[X]$ is the autocorrelation matrix of the available input time samples. Thus, $[X]^T[X]$ is a symmetric positive semi-definite matrix. In fact $[X]^T[X]$ is a Toplitz matrix and therefore is invertible by means of a recursive procedure requiring L^2 operations. The inversion of $[X]^T[X]$ is numerically highly unstable as has been pointed out by Ekstrom [19]. He suggested a Singular Value Decomposition technique for the inversion. In this procedure $[X]^T[X]$ is converted to a diagonal matrix by a similarity transformation. The eigenvalues are then filtered (i.e. eigenvalues below a certain small amount are set to zero). The whole process of premultiplying by $[X]^T$ and then obtaining the singular value decomposition is highly time consuming. In this paper an alternate approach is suggested based upon the conjugate gradient method [20].

With the conjugate gradient method it is not necessary to form the matrix $[X]^T[X]$. The matrix equation $[X][H] = [Y']$ is solved by the following procedure:

By using any initial guess $[H]_0$ for the solution, the matrices $[R]_0$ and $[P]_0$ are generated where

$$[R]_0 = [X][H]_0 - [Y'] \quad (34)$$

and

$$[P]_0 = -[X]^T[R]. \quad (35)$$

The $(n+2)^{th}$ estimate is then given by

$$[H]_{n+1} = [H]_n + t_n[P]_n \quad (36)$$

where

$$t_n = \frac{|[X]^T[R]_n|^2}{|[X][P]_n|^2} \quad (37)$$

and $[R]_n$ and $[P]_n$ are obtained according to the relations

$$[R]_n = [R]_{n-1} + t_{n-1}[X][P]_{n-1} \tag{38}$$

and

$$[P]_n = -[X]^T[R]_n + q_{n-1}[P]_{n-1} \tag{39}$$

where

$$q_{n-1} = \frac{|[X]^T[R]_n|^2}{|[X]^T[R]_{n-1}|^2} \, . \tag{40}$$

The advantage of the conjugate gradient method is that it is an iterative scheme which usually yields excellent results within J iterations where J is the number of independent eigenvalues of $[X]^T[X]$. It has been shown that the conjugate gradient method converges to a solution even when $[X]^T[X]$ is singular. For this case the iteration process is terminated once the solution stops converging (i.e. begins to oscillate). This method has been used successfully in the area of image processing for a similar deconvolution problem [21] . However, an analytical expression is available for the total number of iterations beyond which the iterative procedure can be terminated [25].

The error analysis performed in the previous section is now repeated in order to find mean and variance of the solution obtained by the least squares method when measurement errors are introduced. Corresponding to

$$[X][H] = [Y']$$

$[\tilde{H}]$, an estimate of $[H]$, satisfies the matrix equation

$$[X+P][\tilde{H}] = [Y'+Q] \tag{41}$$

where $[P]$ and $[Q]$ account for the measurement errors and are defined as

$$[P] = \begin{bmatrix} p_0 & 0 & 0 & 0 & \cdots \\ p_1 & p_0 & 0 & 0 & \cdots \\ p_2 & p_1 & p_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{42}$$

$$\text{and } [Q] = \begin{bmatrix} q_0 \\ q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_N \end{bmatrix} \tag{43}$$

The elements of [P] and [Q] are assumed to be statistically independent zero mean random variables.

By assuming the measurement errors are small, an estimate of the impulse response is obtained as

$$[\tilde{H}] = \{[X+P]^T[X+P]\}^{-1}[X+P]^T[Y'+Q]$$

$$= \{[X]^T[X]\}^{-1}\{[I] - ([X]^T[X])^{-1}[A]\}\ [X+P]^T[Y'+Q]$$

$$+ \text{ higher order terms} \tag{44}$$

where

$[I]$ = identity matrix, and

$[A] = [X]^T[P] + [P]^T[X] + [P]^T[P].$

The expected value of the time domain impulse response is given by

$$E[\tilde{H}] \approx \{[X]^T[X]\}^{-1}\{[I]-([X]^T[X])^{-1}[\sigma_p^2]\}[X]^T[Y']$$

$$\text{where } [\sigma_p^2] = E\{[P]^T[P]\}\ . \tag{45}$$

Also note that when measurement errors are associated with the input, then there is a bias in the solution which is given by

$$H - E[\tilde{H}] \simeq \{[X]^T[X]\}^{-2}[\sigma_p^2][X]^T[Y'].$$

Similarly the variance of $[\tilde{H}]$ is defined to be a column vector whose elements are the variances of each of the elements in $[\tilde{H}]$. Note that

$$[\tilde{H}] - E[\tilde{H}] = \{[X]^T[X]\}^{-1} \quad [P]^T[Y'] + [X]^T[Q] - ([X]^T[X])^{-1}$$
$$\cdot\{[X]^T[P] + [P]^T[X]\}\ [X]^T[Y']$$
$$+ \text{ higher order terms}$$
$$= [D] + \text{higher order terms} \tag{46}$$

Then

$$\sigma^2[\tilde{H}] = E\begin{bmatrix} D_1^2 \\ D_2^2 \\ \vdots \\ D_N^2 \end{bmatrix} \tag{47}$$

From (44) an estimate of the time domain impulse response can be obtained at each instance through the column matrix $[\tilde{H}]$. The expected value and the variance are obtained from (45) and (47), respectively, through the elements of the column matrices $E[\tilde{H}]$ and $\sigma^2[\tilde{H}]$. Using Tshebyshev's inequality, as in the previous section, the probability of a particular element $[\tilde{H}_j]$ of the matrix $[\tilde{H}]$ lying between $E[\tilde{H}_j] \pm k\sigma[\tilde{H}_j]$ is greater than $(1 - \frac{1}{k^2})$, i.e.

$$P\{|[\tilde{H}_j] - E[\tilde{H}_j]| \leq k\sigma[\tilde{H}_j]\} \geq 1 - \frac{1}{k^2} . \tag{48}$$

Example 3: Consider, once again, the same problem stated in example 2. From the problem statement, (41) becomes

$$\begin{bmatrix} 1.01 & 0 \\ 0.99 & 1.01 \\ 0 & 0.99 \end{bmatrix} \times \begin{bmatrix} \tilde{h}_0 \\ \tilde{h}_1 \end{bmatrix} = \begin{bmatrix} .99 \\ 2.01 \\ 1.0 \end{bmatrix} \tag{49}$$

After premultiplying both sides of the equations by $[X+P]^T$ and inverting $[X+P]^T[X+P]$, the estimate $[\tilde{H}]$ becomes

$$\begin{bmatrix} \tilde{h}_0 \\ \tilde{h}_1 \end{bmatrix} = \begin{bmatrix} .9865 \\ 1.0168 \end{bmatrix} \tag{50}$$

This compares to

$$[H] = \begin{bmatrix} .9802 \\ 1.0293 \end{bmatrix} \quad (51)$$

by the synthetic division method.

In addition, the least squares approach yields

$$E[H] \simeq \{[X]^T[X]\}^{-1} ([I] - \{[X]^T[X]\}^{-1}[\sigma_p^2])[X]^T[Y']$$

$$\simeq \begin{bmatrix} .9999 \\ .9999 \end{bmatrix} \quad (52)$$

and

$$[\hat{H}]-E[\hat{H}] \simeq \frac{1}{3} \begin{bmatrix} -3p_0 + 2q_0 + q_0 - q_2 \\ -3p_1 - q_0 + q_1 + 2q_2 \end{bmatrix}$$

Thus the variance of the estimate is

$$\sigma^2[\hat{H}] = \begin{bmatrix} 1.6667 \\ 1.6667 \end{bmatrix} \sigma^2 = \begin{bmatrix} .0002 \\ .0002 \end{bmatrix} \quad (53)$$

It is seen that (50) is within the bounds dictated by (48) for the special case in which k=3; i.e., the estimate is within the range given by

$$\begin{bmatrix} .9999 \\ .9999 \end{bmatrix} \pm 3 \begin{bmatrix} .0129 \\ .0129 \end{bmatrix} \quad (54)$$

By comparing (50) to (25) we find that the least squares method yields a better estimate than the synthetic division technique for this problem. Also, the variance of the least squares method is much smaller than the variance of the synthetic division method. However, the synthetic division scheme is computationally much simpler than the least squares method and yields excellent results when σ of the measurement error is very small.

7. OPTIMUM INPUTS FOR WHITE NOISE

In most practical cases, the input is known with great accuracy. The significant errors are then associated with measurement of the output. The question then arises, "Given $p(z)\equiv 0$ and $q(z)\neq 0$, which input $x(z)$ minimizes the variances of $\tilde{h}(z)$ in (22) and (47)?"

It is apparent from (21), (22) and (47) that the variances of $\tilde{h}(z)$ are reduced as the amplitude of $x(z)$ is increased. In a practical situation, however, the amplitude of the input may be limited in order not to excite system nonlinearities. Also, the input amplitude may be constrained due to power limitations. Therefore the above question is pursued subject to a constraint on the input amplitude. For convenience, the constraint is chosen to be on the mean square value of $x(z)$, i.e.

$$\phi_{xx}(0) = \text{autocorrelation of the input } x \text{ of lag zero}$$

$$= \frac{1}{M+1} \sum_{m=0}^{M} x_m^2$$

Assuming $p(z) = 0$ and $q(z)$ to be a white noise sequence, (47) becomes

$$\sigma^2[\tilde{H}] = \sigma_q^{\,2}\{[X]^T[X]\}^{-1}. \tag{56}$$

Observe that $[X]^T[X]$ is a symmetric positive definite matrix with each element along the principal diagonal having the value $(M+1)\phi_{xx}(0)$. Using Levin's results [24], the variance of (56) is minimized if and only if

$$[X]^T[X] = (M+1)\phi_{xx}(0) \cdot [I]$$

where $[I]$ is the identity matrix. Hence, provided the input is deterministic, it follows that the solution for $x(z)$ satisfies

$$\phi_{xx}(0) \neq 0$$
$$\phi_{xx}(\ell) = 0 \quad \text{for } 0 < \ell \leq L \tag{58}$$

Hence, the input sequence $x(z)$ which minimizes the variance is white over a range of L samples.

8. MINIMUM VARIANCE ESTIMATE OF THE IMPULSE RESPONSE FOR COLORED NOISE

For the case in which the noise $q(z)$ is not white, as was assumed in the preceding section, then Laplace [22], Gauss [23] and Markov [24] have shown that the minimum variance unbiased estimate for the impulse response is given by

$$[X]^T[\phi_{qq}]^{-1}[X][\tilde{H}_M] = [X]^T[\phi_{qq}]^{-1}[Y'] \tag{59}$$

Observe the similarity to the solution given by (32). In (59) the matrix $[\phi_{qq}]$ represents the covariance matrix of the noise sequence $q(z)$. The subscript M in $[\tilde{H}_M]$ implies they are Markov estimates.

Equation (61) can be solved in a very computationally efficient way. Since $[\phi_{qq}]$ is a non-singular symmetric covariance matrix, its inverse can always be factored into the following form

$$[\phi_{qq}]^{-1} = [W]^T[W] \tag{60}$$

By application of (60), (59) can then be written as

$$[C]^T[C][\tilde{H}_M] = [C]^T[W][Y'] \tag{61}$$

where

$$[C] \triangleq [W][X] \tag{62}$$

Equation (62) can now be solved by the conjugate gradient method as outlined in section 5.

If the noise is white then (59) is equal to (32) and the solutions are identical. If the noise $q(z)$ is not white, the Markov estimates $[\tilde{H}_M]$ are much more difficult to compute. However the disadvantages of doing more computation are offset by the fact that the Markov estimates are the minimum variance unbiased linear estimates of [H] (i.e, linear in [Y']). If the noise sequence q(z) is Gaussian, it can be shown that $[\tilde{H}_M]$ is an unbiased estimate and the Cramer-Rao inequality holds with the equality sign in [25].

9. CONCLUSION

Two techniques have been presented to compute the time domain impulse response from measured time domain input and output waveforms. The computation is carried out directly in the time domain without the need to perform any Laplace Transforms. Error bounds are presented when the measured waveforms are contaminated with noise and the statistics of the measurement errors are known. The input required under an amplitude constraint to minimize the variance of the impulse response estimate is also discussed. Finally, a generalized least squares technique is presented which yields a minimum variance unbiased estimate for the impulse response.

Work is currently under way dealing with numerical examples from measured waveforms.

ACKNOWLEDGEMENT: Grateful acknowledgement is made to Dr. Henry Mullaney for his encouragement and interest in this work. Thanks are also due to the reviewers who significantly improved the presentation of this paper.

REFERENCES

[1] E.M. Kennaugh and D.L. Moffatt, "Transient and Impulse Response Approximations," Proc. IEEE, Vol. 53, Aug. 1965, pp. 893-901.

[2] D.L. Moffatt and R.K. Mains, "Detection and Discrimination of Radar Targets," IEEE Trans. on AP., Vol AP-23, No. 3, May 1975, pp. 358-367.

[3] A.A. Ksienski, Y. Lin and L.J. White, "Low Frequency Approach to Target Identification," Proc. IEEE, Vol. 63, No. 12, Dec. 1975, pp. 1651-1660.

[4] J.D. Young, "Radar Imaging from Ramp Response Signatures," IEEE Trans. on AP, Vol. 24, No. 3, May 1976, pp. 276-282.

[5] L.C. Chan, D.L. Moffatt and L. Peters, "A Characterization of Subsurface Radar Targets," Proc. IEEE, Vol. 67, No. 7, July 1979, pp. 991-1000.

[6] C.W. Chuang and D.L. Moffatt, "Natural Resonances of Radar Targets via Prony's Method and Target Discrimination," IEEE Trans. on Aerospace and Electronics, Vol. AES-12, No. 5, Sept. 1976, pp. 583-589.

[7] J.N. Brittingham, E.M. Miller and J.L. Willows, "Pole Extraction from Real Frequency Information," Proc. IEEE, Vol. 68, No. 2, Feb. 1980, pp. 263-273.

[8] M. Van Blaricum and R. Mittra, "Techniques for Extracting the Complex Resonances of a System Directly From its Transient Response," Interaction Note 301, December 1975. (Also in IEEE Trans. on A&P, Vol. AP-23, No. 6, Nov. 1975).

[9] C.L. Bennett, R. Hieronymus and H. Mieras, "Impulse Response Target Study," Final Report on F30602-76-C-0209, Sperry Research Center, Sudbury, Mass.

[10] V.K. Jain, "Filter Analysis by Use of Pencil of Functions: Part I", IEEE Trans. on Circuits and Systems, Vol. CAS-21, No. 5, September 1974.

[11] V.K. Jain, "Decoupled Method for Approximation of Signals by Exponentials," IEEE Trans. on System Science and Cybernetics, July 1970.

[12] T.K. Sarkar, J. Nebat, D.D. Weiner and V.K. Jain, "Suboptimal Approximation/Identification of Transient Waveforms from Electromagnetic Systems by Pencil of Function Method." IEEE Trans. on Ant. and Propagat. Nov. 1980.

[13] T.K. Sarkar, et. al., "A Comparison of the Pencil of Function Method with Prony's Method, Wiener Filters and Other Identification Techniques," TR II on Contract F33615-77-C-2059, Dec. 1977.

[14] T.K. Sarkar, et. al., "Use of First Order Filters in Pencil of Function Method," TR3 on Contract N00014-79-C-0598, Rochester, New York, 1980.

[15] F. Ba Hli, "A Time Domain Approach," in "Aspects of Network and System Theory," edited by R. E. Kalman and N. Declaris, Holt Rinehart & Winston, New York, 1971.

[16] F. Ba Hli, "Network Synthesis by Impulse Response for Specified Input and Output in Time Domain," Ph.D. Thesis, MIT, September, 1953.

[17] A. Papoulis, "Probability, Random Variables and Stochastic Processes," McGraw Hill, 1965; pp. 150 and 259.

[18] E.A. Robinson and M.T. Silvia, "Digital Foundations of Time Series Analysis - The Box-Jenkins Approach," Holden Day, 1979.

[19] M.P. Ekstrom, "A Spectral Characterization of the Ill-Conditioning in Numerical Convolution," IEEE Trans. on Acoustics, Speech and Signal Processing, Aug. 1973, pp. 344-347.

[20] T.K. Sarkar, K. Siarkiewicz and R. Stratton, "Survey of Numerical Methods for Solution of Large Systems of Linear Equations for Electromagnetic Field Problems," Technical Report, Dec. 1979.

[21] C. Johnson, "Numerical Solution of Ill-Posed Problems Using the Conjugate Gradient Method," Int. Symposium on Ill-Posed Problems: Theory and Practice, Oct. 1979.

[22] P.S. Laplace, "Theorie Analytique des Probabilities" 3rd Edition, Oeuvres, Paris, 1886.

[23] C.F.Gauss, "Methode des Moindres Carres (translated by J. Bertrand), Paris, 1855.

[24] M.J. Levin, "Estimation of the Characteristics of Linear Systems in the Presence of Noise," D. Sc. Thesis, Columbia University, New York, 1959.

[25] O.M. Alifanov and S.V. Rumyantsev, "Regularizing Gradient Algorithms for Inverse Thermal Conduction Problem," Inzhenerno-Fizicheskii Zhurnal, Vol. 39, No. 2, pp. 253-258, August 1980.